

# Analysis of Packet Loss Processes in High-Speed Networks

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**Abstract**—The packet loss process in a single server queueing system with a finite buffer capacity is analyzed. The model used addresses the packet loss probabilities for packets within a block of consecutive sequence of packets. In contrast to other work that used an independence assumption to compute the loss probabilities of packets within a block, an analytical approach is presented that yields efficient recursions for the computation of the distribution of the number of lost packets within a block of packets of fixed or variable size for several arrival models and several number of sessions. Numerical examples are provided to compare the distribution obtained from our analysis with the distribution obtained by using the independence assumption. The results give insight to the following areas related to high-speed networks: 1) forward error correction schemes become less efficient due to the bursty nature of the packet loss processes; 2) real time traffic such as voice and video might be more sensitive to network congestion than was previously assumed; 3) the retransmission probability of ATM messages has been over-estimated by the use of independence assumption.

**Index Terms**— Packet loss processes, blocking probability, finite-queues, high-speed networks, ATM, forward error recovery.

## I. INTRODUCTION

**F**AST PACKET switching (or its variants like ATM) are now broadly accepted as the universal technique for constructing high-speed multimedia communication networks ([2], [4], [10], [16]). Due to the high throughput demands and the mixed traffic environment, these networks usually employ simplified and universal congestion control mechanisms which are based on call bandwidth reservation, input rate enforcement, and packet discard at the intermediate nodes. Simple *open loop* congestion control mechanisms which use knowledge of the extrinsic parameters associated with the connection and control the source by forcing it to conform with these parameters, has been suggested for these networks. The leaky bucket scheme proposed in [16] and the schemes used in PARIS [4] and in [1] are examples of such mechanisms. Packet discarding is unavoidable during temporary overload situations due to buffer overflow. Packets may be lost also due to bit errors. However, current high-speed networks which use fiber-optic transmission links can achieve very low bit-error rates ( $10^{-14}$  error rates can be achieved over long-distance fibers) that makes it negligible compared to loss due to congestion.

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Therefore, in this paper we shall concentrate on the study of packet loss processes due to buffer overflow.

Understanding the packet loss behavior is crucial for the proper design of the real time (e.g., voice and video) coding and playback mechanisms and the methods of congestion control and error recovery for data. It is also essential for sizing the various buffers at the transmission links and switching elements. Individual packet loss probabilities (where in our terms a packet is the integral transmitted quantity) are usually not sufficient for proper understanding of the system behavior. In general, packets are more likely to be rejected because of buffer overflow as their rate of arrival to the buffer increases. Also, if a packet has been rejected, then it is more likely that consecutive packets will also be rejected<sup>1</sup>. Consequently, it is clear that there is a strong correlation between consecutive packet losses, and therefore losses are bursty. It is well known that the bursty nature of the packet loss process can effectively reduce the service quality, especially for sources which are sensitive to long bursts of losses (such as voice, video, and some error recovery techniques employed for data). It is less known that some sources may actually benefit from the bursty nature of the loss process. For instance, sources that transmit *messages* that are composed of several *packets* may need less retransmissions of messages when the loss process is bursty. The goal of this paper is the study of packet loss processes in systems that can accommodate a finite number of packets.

An important design issue in high-speed networks is the scheme of packet loss recovery. Forward packet recovery schemes for high-speed networks have been investigated recently for both data and video services, see, e.g., [10], [14], [15]. In these schemes, it has been proposed to group data packets into blocks of predetermined size, and add to each block a number of parity packets which contain redundant information. The number of parity packets and their construction determine the maximum number of lost packets in a block that can be tolerated. Below this number, all missing packets can be recovered using the redundant information. For such recovery schemes, it is desirable to derive the probability distribution of the *number* of packets that may be lost within a block. As will be demonstrated later, such forward error recovery schemes become much less efficient as the packet loss process becomes more bursty.

<sup>1</sup>It can be shown for the  $M/M/1/K$  system for example, that this holds for all  $K \geq 2$ , i.e., the conditional probability of rejection given a prior rejection is strictly higher than the unconditional probability of rejection (for an arbitrary arrival).

Another application in which it is important to know the probability distribution of the number of packets lost within a block is related to the asynchronous transfer mode (ATM) emerging standard [2]. In ATM, the current status of the proposed error recovery scheme calls for error recovery of lost cells (ATM packets) using a retransmission at the message (or block) level (above the adaptation layer). This is due to the fact that individual cells of a message are not numbered. This strategy increases the effective loss probability (or retransmission probability) for messages as compared to networks which keep the error recovery units to be the same as the transmission units (i.e., PARIS [4]). This is because any cell loss within the message results in a retransmission of the entire message [3]. On the other hand, the correlation of loss (or no-loss) among the cells of the same message plays a positive role in such a scheme. We shall show that evaluating the message retransmission probability using an independence loss assumption is quite pessimistic.

The model we use in this paper for ascertaining the correlation in the packet loss process consists of a source that generates packets and sends them through a single server with a finite number of buffers, which represents the network. We analyze the packet loss process. In particular, we introduce an efficient recursion to obtain the distribution of the number of lost packets in a block of arrivals of a given size for different arrival models and different number of sessions. Earlier studies that considered similar problems ([10], [14], [15]) used an *independence assumption*, i.e., the assumption that the event of packet loss is independent from packet to packet, and the loss probability of every packet is the same (i.i.d. losses). This assumption can lead to erroneous conclusions, as was first observed in [15] by comparing these results to those obtained from simulations. In this paper, we compare the distribution obtained from our analysis with that obtained from the independence assumption. Numerical examples are provided to show that the distribution we compute may be worse compared to the distribution computed from the independence assumption for applications such as forward error correction or better for applications such as straight message retransmission.

The paper is structured as follows: In Section II, we focus on continuous time systems and a fixed block size (counted in packets). The continuous time model is suitable for the analysis of variable size packet systems. We first present the analysis of a single session with Poisson arrivals, and discuss the numerical results for some examples. Then we proceed to the analysis of a binary Markov (bursty) arrival process. In addition, we extend the previous results to the case of multiple session multiplexing and obtain the distribution of the number of packets lost in a given block of arrivals that belong to a particular session. In Section III, we analyze the single session model of Section II for the case of variable block size. Section IV addresses the discrete time system which better describes an ATM based system. Numerical results are also obtained for this case.

Note that the order by which we present the results does not necessarily reflect their relative practical significance. It was mainly chosen in order to facilitate the technical presentation and to improve understanding.

## II. CONTINUOUS TIME SYSTEMS: FIXED BLOCK SIZE

In this section we consider systems with variable length packets whose transmission time is exponentially distributed with parameter  $\mu$ . The packets are stored in a queue that can accommodate up to  $M$  packets and are served (transmitted) according to an arbitrary policy. If a packet arrives at a system that contains  $M$  packets, it is lost. The packets are grouped into fixed size blocks, namely, every  $n$  consecutive packets form a *block* and we are interested in the probability distribution of the number of lost packets within a block in steady-state. We consider systems with a single arrival stream (*single session*) and with several arrival streams (*multiple sessions*). For both systems we investigate the loss probability distribution for Poisson traffic and for bursty traffic.

### A. A Single Session

1) *Poisson Traffic*: Here we assume that packets arrive at the system according to a Poisson process with rate  $\lambda$ . The average load  $\rho$  is defined as  $\rho \triangleq \lambda/\mu$ . We recall that the stationary probability of having  $i$  packets in the system at an arrival epoch (and also at an arbitrary epoch),  $\Pi(i)$ ,  $0 \leq i \leq M$ , is given by (see, e.g., [11, p. 104]),

$$\Pi(i) = \rho^i / \sum_{l=0}^M \rho^l, \quad 0 \leq i \leq M. \quad (1)$$

Our purpose in this section is to compute the probabilities  $P(j, n)$ ,  $n \geq 1$ ,  $0 \leq j \leq n$  of  $j$  losses in a block of  $n$  consecutive packets. We carry the computation by conditioning on the number of packets seen in the system by the first packet of the block when it arrives. To that end we define  $P_i^a(j, n)$ ,  $i = 0, 1, \dots, M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , to be the probability of  $j$  losses in a block of  $n$  packets, given that there are  $i$  packets in the system just before the arrival epoch of the first packet in the block. Since the first packet in a block is arbitrary,

$$P(j, n) = \sum_{i=0}^M \Pi(i) P_i^a(j, n). \quad (2)$$

To complete the computation we need to compute the probabilities  $P_i^a(j, n)$ ,  $i = 0, 1, \dots, M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ . To that end we will introduce recurrence relations. To define the recursion we need the quantity  $Q_i(k)$ ,  $i = 0, 1, \dots, M$ ,  $0 \leq k \leq i$ , which is the probability that  $k$  packets out of  $i$  leave the system (are transmitted) during an inter-arrival period. Note that this probability is equivalent to the probability of  $k$  Poisson arrivals with rate  $\mu$  during a period exponentially distributed with rate  $\lambda$ , with the restriction that no more than  $i$  arrivals occur during a period. We have (see, e.g., [13]),

$$Q_i(k) = \rho \left( \frac{1}{1+\rho} \right)^{k+1}, \quad 0 \leq k \leq i-1, \\ Q_i(i) = \left( \frac{1}{1+\rho} \right)^i. \quad (3)$$

TABLE I  
PROBABILITY OF PACKET LOSS WITH  $M = 20$  AND  $n = 10$

$j$	$\rho = 0.8$		$\rho = 1.0$		$\rho = 1.5$	
	$P(j, n)$	$P_{\text{ind}}(j, n)$	$P(j, n)$	$P_{\text{ind}}(j, n)$	$P(j, n)$	$P_{\text{ind}}(j, n)$
0	$9.90 \cdot 10^{-1}$	$9.77 \cdot 10^{-1}$	$8.32 \cdot 10^{-1}$	$6.14 \cdot 10^{-1}$	$1.94 \cdot 10^{-1}$	$1.73 \cdot 10^{-2}$
1	$3.86 \cdot 10^{-3}$	$2.28 \cdot 10^{-2}$	$4.76 \cdot 10^{-2}$	$3.07 \cdot 10^{-1}$	$9.71 \cdot 10^{-2}$	$8.66 \cdot 10^{-2}$
2	$2.55 \cdot 10^{-3}$	$2.39 \cdot 10^{-4}$	$3.88 \cdot 10^{-2}$	$6.91 \cdot 10^{-2}$	$1.15 \cdot 10^{-1}$	$1.95 \cdot 10^{-1}$
3	$1.62 \cdot 10^{-3}$	$1.49 \cdot 10^{-6}$	$2.99 \cdot 10^{-2}$	$9.21 \cdot 10^{-3}$	$1.27 \cdot 10^{-1}$	$2.60 \cdot 10^{-1}$
4	$9.69 \cdot 10^{-4}$	$6.07 \cdot 10^{-9}$	$2.16 \cdot 10^{-2}$	$8.06 \cdot 10^{-4}$	$1.29 \cdot 10^{-1}$	$2.28 \cdot 10^{-1}$
5	$5.37 \cdot 10^{-4}$	$1.70 \cdot 10^{-11}$	$1.44 \cdot 10^{-2}$	$4.83 \cdot 10^{-5}$	$1.18 \cdot 10^{-1}$	$1.37 \cdot 10^{-1}$
6	$2.69 \cdot 10^{-4}$	$3.31 \cdot 10^{-14}$	$8.55 \cdot 10^{-3}$	$2.01 \cdot 10^{-6}$	$9.63 \cdot 10^{-2}$	$5.69 \cdot 10^{-2}$
7	$1.17 \cdot 10^{-4}$	$4.41 \cdot 10^{-17}$	$4.39 \cdot 10^{-3}$	$5.75 \cdot 10^{-8}$	$6.71 \cdot 10^{-2}$	$1.63 \cdot 10^{-2}$
8	$4.19 \cdot 10^{-5}$	$3.85 \cdot 10^{-20}$	$1.84 \cdot 10^{-3}$	$1.08 \cdot 10^{-9}$	$3.76 \cdot 10^{-2}$	$3.05 \cdot 10^{-3}$
9	$1.09 \cdot 10^{-5}$	$2.00 \cdot 10^{-23}$	$5.58 \cdot 10^{-4}$	$1.20 \cdot 10^{-11}$	$1.52 \cdot 10^{-2}$	$3.39 \cdot 10^{-4}$
10	$1.57 \cdot 10^{-6}$	$4.66 \cdot 10^{-27}$	$9.30 \cdot 10^{-5}$	$6.00 \cdot 10^{-14}$	$3.36 \cdot 10^{-3}$	$1.70 \cdot 10^{-5}$

The recursion for computing the quantities  $P_i^a(j, n)$  is initiated for  $n = 1$  with the following obvious relations:

$$P_i^a(j, 1) = \begin{cases} 1, & j = 0, \\ 0, & j \geq 1, \end{cases} \quad i = 0, 1, \dots, M-1, \quad (4)$$

and for  $i = M$ , we have

$$P_M^a(j, 1) = \begin{cases} 1, & j = 1, \\ 0, & j = 0, \quad j \geq 2. \end{cases} \quad (5)$$

For  $n \geq 2$ , we have the following recursive equations:

$$P_i^a(j, n) = \sum_{k=0}^{i+1} Q_{i+1}(k) P_{i+1-k}^a(j, n-1), \quad 0 \leq i \leq M-1, \\ P_M^a(j, n) = \sum_{k=0}^M Q_M(k) P_{M-k}^a(j-1, n-1). \quad (6)$$

The explanation of (6) is clear. When the first packet of a block arrives and sees  $i$  ( $0 \leq i \leq M-1$ ) packets in the system, it is not lost. To have  $j$  lost packets out of the block of size  $n$ ,  $j$  packets must be lost out of the  $n-1$  packets starting from the next arrival epoch that will see  $i+1-k$  packets if  $k$  ( $0 \leq k \leq i+1$ ) packets are served between the arrival epochs of the first packet and the subsequent packet. When the first packet arrives and sees  $M$  packets in the system, it is lost, so we have already counted one loss. Hence, to have  $j$  losses out of the block of size  $n$ ,  $j-1$  packets must be lost out of the  $n-1$  packets starting from the next arrival epoch that will see  $M-k$  packets if  $k$  ( $0 \leq k \leq M$ ) packets are served between the arrival epochs of the first packet and the subsequent packet.

Using the initial conditions (4)–(5) one can compute the probabilities  $P_i^a(j, n)$  for any  $n \geq 1$ , recursively, using (6). The number of simple operations (additions and multiplications) needed for this computation is of the order  $O(M^2 n^2)$ .

An alternative recursion for the computation of  $P(j, n)$  can be obtained as follows. Let  $P_i^{ar}(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , be the probability of  $j$  losses in a block of  $n$  consecutive packets that arrive to the system after an arbitrary time instant, given that there are  $i$  packets in the system at

that time instant. Since the first packet in a block is arbitrary, we have (for  $n \geq 2$ )

$$P(j, n) = \sum_{i=0}^{M-1} \Pi(i) P_{i+1}^{ar}(j, n-1) + \Pi(M) P_M^{ar}(j-1, n-1).$$

We proceed to obtain a recursion for the computation of  $P_i^{ar}(j, n)$ . For  $0 \leq i \leq M-1$ , the recursion is initiated with the same relation as (4). For  $i = M$ , we have

$$P_M^{ar}(j, 1) = \begin{cases} \lambda/(\lambda + \mu), & j = 1, \\ \mu/(\lambda + \mu), & j = 0, \quad j \geq 2. \end{cases} \quad (7)$$

For  $n \geq 2$ , we have the following recursive equations:

$$P_0^{ar}(j, n) = P_1^{ar}(j, n-1), \\ P_i^{ar}(j, n) = \frac{\lambda}{\lambda + \mu} P_{i+1}^{ar}(j, n-1) + \frac{\mu}{\lambda + \mu} P_{i-1}^{ar}(j, n), \\ 1 \leq i \leq M-1,$$

$$P_M^{ar}(j, n) = \frac{\lambda}{\lambda + \mu} P_M^{ar}(j-1, n-1) + \frac{\mu}{\lambda + \mu} P_{M-1}^{ar}(j, n). \quad (8)$$

The explanation of (8) is as follows. In the  $M/M/1$  system, given that the system is not empty, the probability of an arrival of a packet to the system before a departure of a packet from the system is given by  $\lambda/(\lambda + \mu)$ , and the probability of a departure before an arrival is given by  $\mu/(\lambda + \mu)$ . Conditioning on the next event (arrival or departure), the recursions in (8) are obtained.

The procedure for the calculation of  $P_i^{ar}(j, n)$  from (8) proceeds as follows. First, the probabilities  $P_i^{ar}(j, 1)$ ,  $0 \leq i \leq M$ , are computed from the initial conditions. In step  $k$ ,  $k = 1, 2, \dots, n$ , the probabilities  $P_i^{ar}(j, k)$  are computed recursively for each  $i$  in increasing order from (8). Note that, in each step  $k$ , the number of simple operations needed for the computation of  $P_i^{ar}(j, k)$ ,  $0 \leq i \leq M$ , is  $O(M)$  and the overall number of simple operations in this procedure is of the order  $O(Mn^2)$ . Hence, this recursion is more efficient than the one of (6).

The probabilities  $P(j, n)$  ( $0 \leq j \leq n$ ) are given in Table I for a system with  $M = 20$  packets, block of size  $n = 10$  and for different average loads  $\rho = 0.8, 1, 1.5$ . For comparison purposes we also give in the table the quantity  $P_{\text{ind}}(j, n)$  ( $n \geq 1, 0 \leq j \leq n$ ), which represents the probability of  $j$  losses in a block of  $n$  packets under an independence assumption. The commonly used independence assumption assumes that each packet of the block of size  $n$  finds the system full (and hence is lost) with probability  $p = \Pi(M)$  independently of other packets. With this assumption the number of lost packets in a block of size  $n$ ,  $n \geq 1$ , is a random variable, binomially distributed, with parameters  $n, p$ . That is,

$$P_{\text{ind}}(j, n) = \binom{n}{j} p^j (1-p)^{n-j}, \quad 0 \leq j \leq n. \quad (9)$$

Table I gives a clear indication that the independence assumption may yield overly optimistic results. Furthermore, an interesting phenomenon can be noticed from the first row of Table I. Correlation exists not only for packet loss but also for the no-loss process. That is, without forward error correction, the probability of *block* loss under the independence assumption is higher than the exact probability of *block* loss, and hence we may not need to implement forward error-correction scheme in order to achieve a prespecified (low) *block* loss probability.

We conclude this sub-section by introducing an additional important measure of packet loss for real-time packet sessions—the average number of consecutively lost packets, referred to as the session average packet gap, see [6]. Consider an arbitrary packet arriving to the system and define the random variable  $X$  to be the number of *consecutive* lost packets. We use the expectation of  $X$  as a measure for the average packet gap. From the definition of the probability  $P(n, n)$ , we have that  $\text{Prob}\{X \geq n\} = P(n, n)$ ,  $n \geq 1$ . Therefore,

$$E[X] = \sum_{n=1}^{\infty} \text{Prob}\{X \geq n\} = \sum_{n=1}^{\infty} P(n, n). \quad (10)$$

From (6), we have that  $P_M^a(n, n) = Q_M(0)P_M^a(n-1, n-1) = \dots = (Q_M(0))^{n-1}P_M^a(1, 1)$  since to have  $n$  losses out of  $n$  consecutive packets, each arriving packet must arrive at a full buffer. Consequently, using (3) and (5), we have that  $P_M^a(n, n) = [\rho/(1+\rho)]^{n-1}$ . Using (2), we conclude that

$$\begin{aligned} P(n, n) &= \sum_{i=0}^M \Pi(i)P_i^a(n, n) \\ &= \Pi(M)P_M^a(n, n) = \Pi(M) \left( \frac{\rho}{1+\rho} \right)^{n-1}, \quad n \geq 1. \end{aligned} \quad (11)$$

Therefore, from equations (10) and (11) we have  $E[X] = (1+\rho)\Pi(M)$ , and  $E[X]$  increases in  $\rho$ .

2) *Bursty Traffic*: In this section the source is modeled as an *interrupted Poisson process* (IPP). This model is widely used in the literature to represent bursty and correlated cell arrivals, where a source may stay for relatively long durations in active (ON) and silent (OFF) periods [19]. We define the

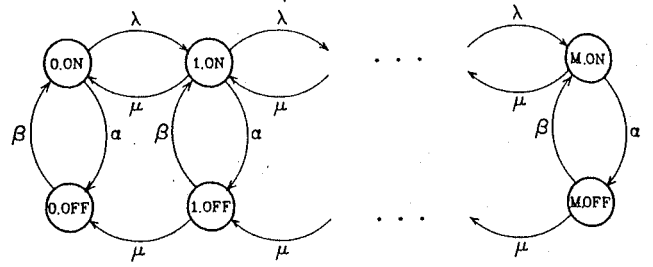


Fig. 1. The state diagram of the bursty traffic, single session system.

“active periods” and the “silent periods” of the source as the time periods during which the source generates packets or is idle, respectively. We assume that packets are generated by the source during active periods according to a Poisson process with rate  $\lambda$ . The duration of the active periods and the silent periods are assumed to be two independent sets of independent and identically distributed random variables exponentially distributed with (positive) parameters  $\alpha$  and  $\beta$ , respectively.

In the following, we derive the probability  $P(j, n)$  of losing  $j$  packets out of a block of length  $n$  for the bursty traffic model. The approach we use is the same as that in Section II-A-1. We first determine the probability that an arrival will see  $i$  packets in the system and the probability of  $k$  out of  $i$  packet transmissions during an interarrival period. Then we use (4)–(6) to obtain  $P(j, n)$ .

Let  $N(t)$  ( $N(t) = 1, \dots, M$ ), be the number of packets in the system at time  $t$  and  $s(t)$  ( $s(t) \in \{\text{ON}, \text{OFF}\}$ ) be the state of the source at time  $t$ . The vector  $(N(t), s(t))$  is a finite-state Markov process. Let  $(N, s) \triangleq \lim_{t \rightarrow \infty} (N(t), s(t))$  be the state of the system in the steady-state regime, and denote its probability by  $\Pi(N, s)$ . The state diagram of the system in steady-state is plotted in Fig. 1. From this figure, we obtain the following equations:

$$(\mu 1\{i \neq 0\} + \beta)\Pi(i, \text{OFF}) = \alpha\Pi(i, \text{ON}) + \mu\Pi(i+1, \text{OFF}), \quad 0 \leq i \leq M, \quad (12)$$

$$\lambda\Pi(i-1, \text{ON}) = \mu\Pi(i, \text{ON}) + \mu\Pi(i, \text{OFF}), \quad 1 \leq i \leq M, \quad (13)$$

where in (12), we define  $\Pi(M+1, \text{OFF}) \triangleq 0$  and  $1\{\cdot\}$  is an indicator function. The state probabilities  $\Pi(i, \text{ON})$ ,  $\Pi(i, \text{OFF})$ ,  $0 \leq i \leq M$ , are obtained from (12) and (13), respectively. First, we assume an initial positive value for the quantity  $\Pi(M, \text{ON})$ , then the quantities  $\Pi(i, \text{OFF})$  and  $\Pi(i, \text{ON})$  for  $i = M, M-1, \dots, 1, 0$ , are obtained recursively from (12) and (13), respectively. Finally, these quantities are normalized, so that  $\sum_{i=0}^M [\Pi(i, \text{ON}) + \Pi(i, \text{OFF})] = 1$ .

Denote by  $\Pi(i|\text{ON})$  the probability of  $i$  packets in the system given that the source is active. Then

$$\Pi(i|\text{ON}) = \frac{\alpha + \beta}{\beta} \Pi(i, \text{ON}), \quad 0 \leq i \leq M.$$

Note that, due to the PASTA (Poisson arrivals see time averages [18]) property, the probability  $\Pi(i|\text{ON})$  corresponds to the probability that an arriving packet finds  $i$  packets in the system.

We turn now to compute the probability  $Q_i^{\text{ON}}(k)$ ,  $0 \leq i \leq M$ ,  $0 \leq k \leq i$ , that  $k$  packets out of  $i$  leave the system (are transmitted) during an inter arrival period in the bursty traffic model. Let,

$$\begin{aligned} \Delta &\triangleq (\lambda + \alpha + \beta)^2 - 4\lambda\beta, \\ \alpha_{1,2} &\triangleq 1 + \frac{\lambda + \alpha + \beta}{2\mu} \pm \frac{\sqrt{\Delta}}{2\mu}, \\ c_1 &\triangleq \frac{\lambda(\beta + \mu) - \lambda\mu\alpha_1}{\alpha_1(\alpha_2 - \alpha_1)}, \quad c_2 \triangleq \frac{\lambda(\beta + \mu) - \lambda\mu\alpha_2}{\alpha_2(\alpha_1 - \alpha_2)}. \end{aligned}$$

Note that by definition  $\Delta > 0$  and  $\alpha_{1,2} > 1$ .

*Proposition 1:* Using the previous definitions, the following holds:

$$\begin{aligned} Q_i^{\text{ON}}(k) &= c_1 \left(\frac{1}{\alpha_1}\right)^k + c_2 \left(\frac{1}{\alpha_2}\right)^k, \quad 0 \leq k \leq i-1. \\ Q_i^{\text{ON}}(i) &= c_1 \frac{(1/\alpha_1)^i}{1-1/\alpha_1} + c_2 \frac{(1/\alpha_2)^i}{1-1/\alpha_2}. \end{aligned} \quad (14)$$

The interested reader is referred to [5] for the proof of this proposition.

The computation of the probability distribution  $P(j, n)$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , in this case continues in the same way as in the Poisson traffic model with the probability distributions  $\Pi(i|\text{ON})$  and  $Q_i^{\text{ON}}(k)$  replacing the corresponding distributions  $\Pi(i)$  and  $Q_i(k)$  in (2) and (6), respectively.

### B. Multiple Sessions

Here we assume that packets arrive to the system from  $S$  independent sources, that is, the inter arrival times and the transmission times of packets from each source are mutually independent. The arrival process from source  $s$ ,  $s = 1, 2, \dots, S$ , is assumed to be Poisson with rate  $\lambda_s$ . The overall arrival process to the system is then Poisson with rate  $\lambda \triangleq \sum_{s=1}^S \lambda_s$ . For the system with Poisson rate  $\lambda$  and exponential transmission rate  $\mu$ , the probabilities  $\Pi(i)$  and  $Q_i(k)$  for  $i = 0, 1, \dots, M$ ,  $0 \leq k \leq i$ , are given in (1) and (3), respectively.

Denote by  $P_i^{s,a}(j, n)$  and  $P_i^{\bar{s},a}(j, n)$ ,  $i = 0, 1, \dots, M$ ,  $s = 1, 2, \dots, S$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , the probabilities of  $j$  losses in a block of  $n$  packets originated from source  $s$ , given that there are  $i$  packets in the system just before the arrival of the first packet in the block, and just before the arrival of a packet from any other source (denoted by  $\bar{s}$ ), respectively. Denote by  $P^s(j, n)$ ,  $s = 1, 2, \dots, S$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , the probability of  $j$  losses in a block of  $n$  packets originated from source  $s$ . Since the first packet in a block is arbitrary, we have that

$$P^s(j, n) = \sum_{i=0}^M \Pi(i) P_i^{s,a}(j, n). \quad (15)$$

We turn now to compute the probabilities  $P_i^{s,a}(j, n)$ ,  $i = 0, 1, \dots, M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , for any source  $s$ ,  $s = 1, 2, \dots, S$ . For  $n = 1$ , (4) and (5) still hold for  $P_i^{s,a}(j, n)$ . The probability of an arrival in the overall Poisson arrival stream being from source  $s$  is equal to  $\lambda_s/\lambda$ . Define  $p(s) \triangleq \lambda_s/\lambda$

and  $p(\bar{s}) \triangleq 1 - p(s)$ . Using the previous definitions, we have (for  $n \geq 2$ )

$$\begin{aligned} P_i^{s,a}(j, n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) \\ &\quad \cdot [p(s)P_{i+1-k}^{s,a}(j, n-1) \\ &\quad + p(\bar{s})P_{i+1-k}^{\bar{s},a}(j, n-1)], \\ &\quad 0 \leq i \leq M-1, \end{aligned}$$

$$\begin{aligned} P_M^{s,a}(j, n) &= \sum_{k=0}^M Q_M(k) \\ &\quad \cdot [p(s)P_{M-k}^{s,a}(j-1, n-1) \\ &\quad + p(\bar{s})P_{M-k}^{\bar{s},a}(j-1, n-1)], \end{aligned} \quad (16)$$

where  $P_i^{\bar{s},a}(j, n)$  for  $n \geq 1$  is given by

$$\begin{aligned} P_i^{\bar{s},a}(j, n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) \\ &\quad \cdot [p(s)P_{i+1-k}^{s,a}(j, n) + p(\bar{s})P_{i+1-k}^{\bar{s},a}(j, n)], \\ &\quad 0 \leq i \leq M-1, \end{aligned}$$

$$P_M^{\bar{s},a}(j, n) = P_{M-1}^{\bar{s},a}(j, n). \quad (17)$$

The procedure for the computation of  $P_i^{s,a}(j, n)$  proceeds as follows. First, the probabilities  $P_i^{s,a}(j, 1)$ ,  $i = 0, 1, \dots, M$ , are computed from the initial conditions (4)–(5). In step  $k$ ,  $k = 1, 2, \dots, n-1$ , the probabilities  $P_i^{\bar{s},a}(j, k)$ ,  $0 \leq i \leq M$ , are computed first from (17) and the probabilities  $P_i^{s,a}(j, k)$ ,  $0 \leq i \leq M$ , (which have been computed in step  $k-1$ ). Then, the probabilities  $P_i^{s,a}(j, k+1)$  are computed recursively from (16). The probabilities  $P_i^{\bar{s},a}(j, k)$ ,  $0 \leq i \leq M-1$ , can be computed recursively from (17) using the method developed in Appendix A. An alternative recursion at arbitrary epochs can be obtained in a similar way to the single session system.

The analysis of the bursty traffic model for the multiple session system is more complicated and therefore omitted. The interested reader is referred to Appendix D in [5] for this analysis.

The probabilities  $P^s(j, n)$  ( $0 \leq j \leq n$ ), are given in Table II for a system with  $M = 20$  packets, block size  $n = 10$ , overall average load  $\rho = 0.8$  ( $\rho \triangleq \lambda/\mu$ ) and for different average loads  $\rho_s = 0.1, 0.4, 0.7$  ( $\rho_s \triangleq \lambda_s/\mu$ ). For comparison purposes, we also include in the table the quantity  $P_{\text{ind}}^s(j, n)$ ,  $0 \leq j \leq n$ , which represents the probability of  $j$  losses in a block of  $n$  packets originated from source  $s$  under an independence assumption (see (9)).

From Table II, it is clear that the differences between the distributions  $P^s(j, n)$  and  $P_{\text{ind}}^s(j, n)$ ,  $0 \leq j \leq n$ , depend on the average load  $\rho_s$ . To further explore these differences, we define the *distance* between these probability distributions as the well known *divergence* [12],

$$D(P^s \| P_{\text{ind}}^s) \triangleq \sum_{j=0}^n P(j, n) L_n \frac{P(j, n)}{P_{\text{ind}}^s(j, n)}, \quad n \geq 1.$$

TABLE II  
PROBABILITY OF PACKET LOSS WITH  $M = 20$ ,  $n = 10$  AND  $\rho = 0.8$

$j$	$P^s(j, n)$			$P_{\text{ind}}^s(j, n)$
	$\rho_s = 0.1$	$\rho_s = 0.4$	$\rho_s = 0.7$	$\rho_s = 0.1, 0.4, 0.7$
0	$9.826 \cdot 10^{-1}$	$9.877 \cdot 10^{-1}$	$9.896 \cdot 10^{-1}$	$9.770 \cdot 10^{-1}$
1	$1.290 \cdot 10^{-2}$	$6.198 \cdot 10^{-3}$	$4.244 \cdot 10^{-3}$	$2.280 \cdot 10^{-2}$
2	$3.394 \cdot 10^{-3}$	$3.313 \cdot 10^{-3}$	$2.718 \cdot 10^{-3}$	$2.392 \cdot 10^{-4}$
3	$8.449 \cdot 10^{-4}$	$1.670 \cdot 10^{-3}$	$1.658 \cdot 10^{-3}$	$1.488 \cdot 10^{-6}$
4	$1.959 \cdot 10^{-4}$	$7.882 \cdot 10^{-4}$	$9.539 \cdot 10^{-4}$	$6.075 \cdot 10^{-9}$
5	$4.141 \cdot 10^{-5}$	$3.422 \cdot 10^{-4}$	$5.080 \cdot 10^{-4}$	$1.701 \cdot 10^{-11}$
6	$7.755 \cdot 10^{-6}$	$1.337 \cdot 10^{-4}$	$2.444 \cdot 10^{-4}$	$3.306 \cdot 10^{-14}$
7	$1.235 \cdot 10^{-6}$	$4.557 \cdot 10^{-5}$	$1.026 \cdot 10^{-4}$	$4.407 \cdot 10^{-17}$
8	$1.570 \cdot 10^{-7}$	$1.295 \cdot 10^{-5}$	$3.555 \cdot 10^{-5}$	$3.855 \cdot 10^{-20}$
9	$1.419 \cdot 10^{-8}$	$2.834 \cdot 10^{-6}$	$9.233 \cdot 10^{-6}$	$1.998 \cdot 10^{-23}$
10	$6.857 \cdot 10^{-10}$	$3.887 \cdot 10^{-7}$	$1.422 \cdot 10^{-6}$	$4.662 \cdot 10^{-27}$

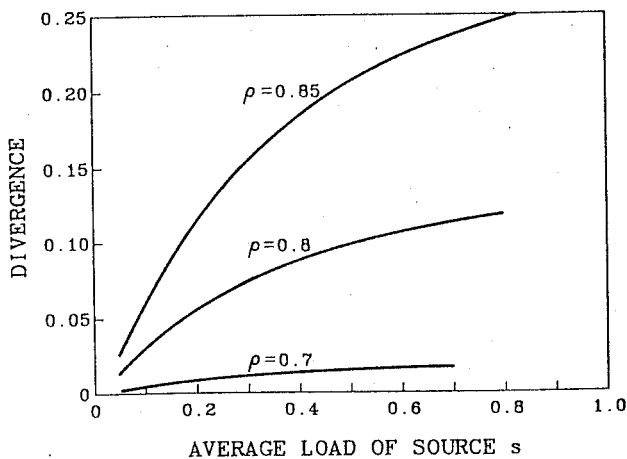


Fig. 2. The divergence  $D(P^s \| P_{\text{ind}}^s)$  as a function of  $\rho_s$  for  $M = 20$ ,  $n = 20$ .

The divergence is plotted in Fig. 2 as a function of the average load  $\rho_s$ , for a system that can accommodate  $M = 20$ , block size  $n = 20$  and overall average loads  $\rho = 0.7, 0.8, 0.85$ .

From Fig. 2, the probability distribution  $P^s(j, n)$  is close to the probability distribution  $P_{\text{ind}}^s(j, n)$  as  $\rho_s$  decreases, for constant average load  $\rho$ . That is, the correlation between lost packets from source  $s$  decreases as the rate of source  $s$  decreases, for a given arrival rate to the system.

### III. CONTINUOUS TIME SYSTEMS: VARIABLE BLOCKS SIZE

In this section, we consider systems with variable length blocks, i.e., arriving packets belong to blocks of lengths that are independent and geometrically distributed with parameter  $q$ . Variable block size (or message size) is typical in data applications where the block can be a document, an e-mail message, or an arbitrary file. This model also assumes a variable size packet which may correspond to some natural partition of the message (i.e., sections of a document, paragraphs of the e-mail message, etc.). We confine ourselves in this section to the analysis of a single session system with Poisson arrival rate  $\lambda$  and transmission time exponentially distributed with rate  $\mu$ . The extension of the analysis to multiple session system

and for a bursty traffic model is similar to the extensions in Section II and is not presented here.

Denote by  $P^b(j)$ ,  $j \geq 0$ , the probability of  $j$  losses in a block in steady-state, and denote by  $P_i^{b,a}(j)$ ,  $i = 0, 1, \dots, M$ ,  $j \geq 0$ , the probability of  $j$  losses in a block following and including an arrival which finds  $i$  packets in the system. We have that

$$P^b(j) = \sum_{i=0}^M \Pi(i) P_i^{b,a}(j), \quad j \geq 0, \quad (18)$$

where the stationary probabilities  $\Pi(i)$ ,  $0 \leq i \leq M$ , are given in (1).

To complete the computation we need to compute the probabilities  $P_i^{b,a}(j)$ ,  $j \geq 0$ . Define  $\bar{q} = 1 - q$ , then for each  $j$ ,  $j \geq 0$ , these probabilities are computed from the following set of equations:

$$P_i^{b,a}(j) = q \mathbf{1}\{j=0\} + \bar{q} \sum_{k=0}^{i+1} Q_{i+1}(k) P_{i+1-k}^{b,a}(j), \quad 0 \leq i \leq M-1, \quad (19)$$

$$P_M^{b,a}(0) = 0, \quad P_M^{b,a}(j) = q \mathbf{1}\{j=1\} + \bar{q} \sum_{k=0}^M Q_M(k) P_{M-k}^{b,a}(j-1), \quad j \geq 1, \quad (20)$$

where the probabilities  $Q_i(k)$ ,  $0 \leq k \leq i$ , where computed in (3) and  $\mathbf{1}\{\cdot\}$  is an indicator function. For each  $j$ ,  $j \geq 0$ , the probability  $P_M^{b,a}(j)$  is first computed recursively from (20), then the probabilities  $P_i^{b,a}(j)$ ,  $0 \leq i \leq M-1$ , can be computed recursively from (19)–(20) using the method presented in Appendix A.

Define the moment generating functions  $F_i^{b,a}(z) \triangleq \sum_{j=0}^{\infty} P_i^{b,a}(j) z^j$ ,  $0 \leq i \leq M$ , and define

$$F^b(z) \triangleq \sum_{j=0}^{\infty} P^b(j) z^j = \sum_{i=0}^M \Pi(i) F_i^{b,a}(z). \quad (21)$$

TABLE III  
PROBABILITY OF BLOCK LOSS WITH  $M = 128$

$q$	$\rho = 0.8$		$\rho = 1.0$		$\rho = 1.5$	
	VBBL System	VPPL System	VBBL System	VPPL System	VBBL System	VPPL System
0.05	$5.098 \cdot 10^{-13}$	$3.603 \cdot 10^{-2}$	$3.467 \cdot 10^{-2}$	$6.317 \cdot 10^{-2}$	$8.054 \cdot 10^{-1}$	$1.576 \cdot 10^{-1}$
0.1	$3.146 \cdot 10^{-13}$	$7.731 \cdot 10^{-3}$	$2.451 \cdot 10^{-2}$	$3.174 \cdot 10^{-2}$	$7.033 \cdot 10^{-1}$	$1.556 \cdot 10^{-1}$
0.2	$2.018 \cdot 10^{-13}$	$5.409 \cdot 10^{-4}$	$1.733 \cdot 10^{-2}$	$1.696 \cdot 10^{-2}$	$5.868 \cdot 10^{-1}$	$1.682 \cdot 10^{-1}$
0.3	$1.574 \cdot 10^{-13}$	$3.871 \cdot 10^{-5}$	$1.415 \cdot 10^{-2}$	$1.231 \cdot 10^{-2}$	$5.177 \cdot 10^{-1}$	$1.831 \cdot 10^{-1}$
0.4	$1.329 \cdot 10^{-13}$	$2.646 \cdot 10^{-6}$	$1.226 \cdot 10^{-2}$	$1.014 \cdot 10^{-2}$	$4.699 \cdot 10^{-1}$	$2.002 \cdot 10^{-1}$
0.5	$1.167 \cdot 10^{-13}$	$1.716 \cdot 10^{-7}$	$1.096 \cdot 10^{-2}$	$8.977 \cdot 10^{-3}$	$4.343 \cdot 10^{-1}$	$2.194 \cdot 10^{-1}$
0.6	$1.050 \cdot 10^{-13}$	$1.054 \cdot 10^{-8}$	$1.001 \cdot 10^{-2}$	$8.331 \cdot 10^{-3}$	$4.061 \cdot 10^{-1}$	$2.407 \cdot 10^{-1}$
0.7	$9.614 \cdot 10^{-14}$	$6.099 \cdot 10^{-10}$	$9.265 \cdot 10^{-3}$	$7.983 \cdot 10^{-3}$	$3.832 \cdot 10^{-1}$	$2.634 \cdot 10^{-1}$
0.8	$8.926 \cdot 10^{-14}$	$3.313 \cdot 10^{-11}$	$8.667 \cdot 10^{-3}$	$7.819 \cdot 10^{-3}$	$3.640 \cdot 10^{-1}$	$2.869 \cdot 10^{-1}$
0.9	$8.349 \cdot 10^{-14}$	$1.679 \cdot 10^{-12}$	$8.171 \cdot 10^{-3}$	$7.760 \cdot 10^{-3}$	$3.476 \cdot 10^{-1}$	$3.105 \cdot 10^{-1}$
0.95	$8.105 \cdot 10^{-14}$	$3.674 \cdot 10^{-13}$	$7.954 \cdot 10^{-3}$	$7.753 \cdot 10^{-3}$	$3.402 \cdot 10^{-1}$	$3.220 \cdot 10^{-1}$

Then, from (19), we have that

$$F_i^{b,a}(z) = q + \bar{q} \sum_{k=0}^{i+1} Q_{i+1}(k) F_{i+1-k}^{b,a}(z), \quad 0 \leq i \leq M-1,$$

$$F_M^{b,a}(z) = qz + \bar{q}z \sum_{k=0}^M Q_M(k) F_{M-k}^{b,a}(z). \quad (22)$$

From (22), a set of  $M+1$  linear equations for the computation of the moments of the generating functions  $F_i^{b,a}(z)$  can be obtained. These equations can be solved recursively using the method in Appendix A. Then, any moment of the number of packet losses in a block can be obtained from (21).

*Numerical Examples:* In some networks such as the asynchronous transfer mode (ATM) standard [2], a large data unit is partitioned to smaller pieces that are sent separately. The use of small transmission units translates into smaller buffers at the intermediate nodes and thus decreases memory requirements (we will ignore the extra overhead imposed by duplicating the headers in each packet). However, since packets (cells) are not individually numbered, the proposed error recovery scheme calls for error recovery at the message (block) level, i.e., any packet (cell) loss within the message results in a retransmission of the entire message. This in turn, increases the loss probability of messages [3]. Other networks use larger packets (i.e., packet  $\equiv$  message) but keep the error recovery units to be the same as the transmission units (i.e., at the packet level). This increases memory requirements due to the larger packets, but reduces loss probability by using a more efficient error recovery scheme.

For systems which use error recovery at the block level, the loss probability of a block is given by  $1 - P^b(0)$  (without forward error correction). We shall refer to these systems as VBBL (variable length blocks with block level error recovery) systems. In what follows we compare these systems with the latter systems referred as VPPL (variable length packets with packet level error recovery) systems using the same amount of buffer memory (but ignoring the extra overhead imposed in VBBL systems). For VBBL systems, we consider the same model presented in this section. For VPPL systems we develop an equivalent model in the following manner: blocks

arrive to the system according to a Poisson process with rate  $q\lambda$ . The number of packets in each block is geometrically distributed with parameter  $q$ . The transmission time of a packet is exponentially distributed with parameter  $\mu$ . Each block is considered as a whole unit (VPPL packet) and join the system only if there is a room in the system for all the packets in the block, otherwise the block is rejected (and lost). Denote by  $\pi(i)$ ,  $0 \leq i \leq M$ , the stationary probability of  $i$  packets in the system, then by straight forward calculation, we have

$$\pi(i) = q\rho(1 - \bar{q}^{M-i+1}) \sum_{k=0}^{i-1} \bar{q}^k \pi(i-1-k), \quad 0 \leq i \leq M. \quad (23)$$

The state probabilities  $\pi(i)$ ,  $0 \leq i \leq M$ , are easily obtained from the recursive equations in (23) and the normalization condition  $\sum_{i=0}^M \pi(i) = 1$ . For the VPPL system, the loss probability of a block is given by

$$\Pr\{\text{block loss in VPPL}\} = \sum_{i=0}^M \Pr\{\text{block size} > M-i\} \pi(i)$$

$$= \sum_{i=0}^M [1 - (1 - \bar{q}^{M-i})] \pi(i).$$

In Table III, we compute the loss probability of a block in the VBBL system and in the VPPL system as a function of the parameter  $q$ , for a system that can accommodate  $M = 128$  packets and for different average loads  $\rho = 0.8, 1, 1.5$  ( $\rho \triangleq \lambda/\mu$ ).

From Table III, we see that for  $\rho = 0.8$ , the VBBL model outperforms the VPPL system for all values of  $q$ , that is the loss probability of a block in the VBBL model is strictly less than it is in the VPPL system and the difference is significant. For  $\rho = 1.5$ , the VPPL system outperforms the VBBL model for all values of  $q$ . For  $\rho = 1$ , the VBBL model outperforms the VPPL system for small values of  $q$  ( $q \leq 0.2$ ) and the VPPL system outperforms the VBBL model for large values of  $q$  ( $q \geq 0.3$ ). Yet, the differences in the two latter cases are small.

## IV. DISCRETE TIME SYSTEMS

In this section, we consider a discrete time model which better describes an ATM based system. In what follows we describe the queueing model and notations used throughout the section. Consider a discrete time queueing system in which the time axis is divided into intervals of equal size, referred to as slots. The slots correspond to the transmission time of a packet, and all packets are assumed to be of the same fixed size. The packets are stored in a buffer that can accommodate up to  $M$  packets, and are served (transmitted) according to an arbitrary policy. Packets arrive randomly to the system from a single source. The arrival process is assumed to be independent and identically distributed from slot to slot. We further assume that at most one packet may arrive instantaneously to the system, and that packets join the system at a first-in-first-join order if there is an empty buffer in the system, otherwise the packet is lost. Let  $b$  be the random variable representing the number of packets that arrive during a slot and define  $b_i \triangleq \Pr\{b = i\}$ ,  $i \geq 0$ . As in the continuous time model, the packets are grouped into fixed size blocks, namely, every  $n$  consecutive packets form a *block*, and we are interested in the probability distribution of the number of lost packets within a block in steady-state. We consider systems with a single arrival stream (single session), where the multiple session system and the bursty traffic model can be analyzed by the same arguments used in the analysis of the continuous time models (Section II).

The system behavior is modeled as a finite-state discrete-time Markov chain, in which the state is the number of packets in the system just before the beginning of a slot. The stationary probability of having  $i$  packets in the system at the beginning of an arbitrary slot,  $\Pi(i)$ ,  $0 \leq i \leq M$ , can be computed recursively from the following set of  $M$  linear equations:

$$\begin{aligned} \Pi(0) &= \Pi(0)b_0 + \Pi(1)b_0, \\ \Pi(i) &= \Pi(0)b_i + \sum_{k=1}^i \Pi(k)b_{i-k+1} + \Pi(i+1)b_0, \\ &\quad 1 \leq i \leq M-2, \\ \Pi(M) &= \Pi(0) \Pr\{b \geq M\}. \end{aligned} \quad (24)$$

Our purpose in this section is to compute the probabilities  $P(j, n)$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , of  $j$  losses in a block of  $n$  packets. We carry the computation by conditioning on the number of packets seen in the system by the first packet in the block when it arrives. To that end we define  $P_i^a(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , to be the probability of  $j$  losses in a block of  $n$ , given that at the beginning of the slot in which the first packet in the block arrives there are  $i$  packets in the system. Consider an arbitrary packet arriving to the system (this packet will be called the *tagged* packet). Since the arrivals are independent and identically distributed from slot to slot, the probability distribution of the number of packets in the system at the beginning of the slot in which this packet arrives is the same as  $\Pi(i)$ . This follows from a discrete time version of the well-known PASTA theorem, see [9]. Since the first packet in a block is arbitrary, the probability  $P(j, n)$  equals to the probability of  $j$  packet losses in the block

of  $n$  packets following the *tagged* packet and including it, and is given by

$$P(j, n) = \sum_{i=0}^M \Pi(i) P_i^a(j, n). \quad (25)$$

To complete the computation we need to compute the probabilities  $P_i^a(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ . In what follows, we shall introduce a recursion for the computation of these probabilities. Note that, the *tagged* packet is more likely to arrive within a large *superpacket* (we call all the packets arriving in a slot a *superpacket*). In any slot, a *superpacket* does not arrive with probability  $b_0$ , and arrives with probability  $1 - b_0$  (i.e., a *Bernoulli arrival process*). We are interested in the number of packets in the superpacket arriving before and after (including) the *tagged* packet, which we denote by the random variables  $b^b$  and  $b^a$ , respectively. Note that  $b^b$  and  $b^a$  are the *backward* and the *residual* recurrence times in the (discrete time) renewal process whose interevent time distribution is given by the random variable  $b$ . Thus, the distributions of  $b^b$  and  $b^a$  are given by

$$\begin{aligned} b_k^b &\triangleq \Pr\{b^b = k\} = \frac{\Pr\{b > k\}}{E[b]}, \quad k = 0, 1, \dots, \\ b_k^a &\triangleq \Pr\{b^a = k\} = \frac{\Pr\{b \geq k\}}{E[b]}, \quad k = 1, 2, \dots \end{aligned}$$

And the joint distribution of  $b^b$  and  $b^a$  is given by,

$$\Pr\{b^b = m, b^a = k\} = \frac{b_{m+k}}{E[b]}, \quad m \geq 0, k \geq 1. \quad (26)$$

To introduce the recursion for the computation of the probability  $P_i^a(j, n)$ , we define  $P_i^d(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , to be the probability of  $j$  losses in a block of  $n$ , starting at the beginning of a slot in which there are  $i$  packets in the system. Now we are ready to introduce the computation when the probabilities  $P_i^d(j, n)$ ,  $0 \leq j \leq n$ , are known. For  $j = 0$ , we have

$$\begin{aligned} P_i^a(0, n) &= \sum_{m=0}^{M-i-1} \left\{ \sum_{k=1}^{M-i-m} \Pr\{b^b = m, b^a = k\} \right. \\ &\quad \cdot P_{(i-1)^+ + m + k}^d(0, n - k) \\ &\quad \left. + 1\{n \leq M - i - m\} \right. \\ &\quad \left. \cdot \Pr\{b^b = m, b^a > n\} \right\}, \\ &\quad 0 \leq i \leq M, \end{aligned} \quad (27)$$

and for  $1 \leq j \leq n$ , we have

$$\begin{aligned} P_i^a(j, n) &= \sum_{m=0}^{M-i-1} \left\{ \sum_{k=1}^{M-i-m} \Pr\{b^b = m, b^a = k\} \cdot P_{(i-1)^+ + m + k}^d(j, n - k) \right. \\ &\quad \left. + 1\{n = M - i - m + j\} \cdot \Pr\{b^b = m, b^a > n\} \right. \\ &\quad \left. + \sum_{k=1}^j \Pr\{b^b = m, b^a = M - i - m + k\} \right. \\ &\quad \left. \cdot P_{M-1\{i \neq 0\}}^d(j - k, n - (M - i - m + k)) \right\} \end{aligned}$$



$$\begin{aligned}
& + \sum_{k=1}^j \Pr\{b^b \geq M-i, b^a = k\} \cdot P_{M-1\{i \neq 0\}}^d(j-k, n-k) \\
& + \mathbf{1}\{n=j\} \Pr\{b^b \geq M-i, b^a > n\}, \\
& \quad 0 \leq i \leq M, \quad (28)
\end{aligned}$$

where an empty sum vanishes and  $n^+ \triangleq \max(0, n)$ . We also define  $P(j, n) = P_i^a(j, n) \triangleq 0$ ,  $n < 0$  or  $j > n$ ,  $P_i^d(j, n) \triangleq 0$ ,  $n < 0$  or  $j > n$ , and  $P_i(0, 0) \triangleq 1$ . The joint probabilities of the random variables  $b^b$  and  $b^a$  in (27) and (28) are obtained by appropriate finite sums of the joint probabilities in (26).

The explanation of (27) is as follows. The first packet of a block (called the *head*) arrives to the system in a slot where  $i$  ( $0 \leq i \leq M$ ) packets are waiting at the beginning of this slot. When the number of packets  $m$  that arrive before the *head* in the same slot (and hence join the system before it) occupies at most  $M-i-1$  empty buffers, the *head* is not lost. When also the number of packets  $k$  that arrive after the *head* (including it) does not occupy the remaining space in the system (i.e.,  $k \leq M-i-m$  packets), then in order to have  $j=0$  packet losses in the block of size  $n$ , there must not be any loss in the next  $n-k$  packet arrivals, starting from the beginning of the next slot in which there are  $(i-1)^+ + m+k$  packets in the system. This completes the explanation of the first term of (27). The second term indicates that, when the block size  $n$  is less than  $M-i-m$  packets, then there are 0 losses in the block independently of the subsequent arrivals. The explanation of (28) is similar. The first three terms represent the case where the *head* finds an empty space and hence is not lost. The explanation of the first term is similar to the explanation of the first term in (27) with  $j, j \geq 1$ , replacing  $j=0$ . The third term represents the case of  $k \leq j$  packet losses in the first slot (the slot in which the *head* arrives). The second term represents the case where the block size  $n$  fits exactly the remaining space of the system  $M-i-m$  and the required number of lost packets  $j$ . Then exactly  $j$  packets out of  $n$  are lost independently of the subsequent packet arrivals. The last two terms represent the case where the *head* finds a full system and hence is lost. The fourth term represents the case of  $k \leq j$  losses in the first slot, and the last term represents the case where the block size  $n$  equals to the required number of losses  $j$ . Then exactly  $j$  packets out of  $j$  are lost independently of the subsequent arrivals.

We now turn to compute the probabilities  $P_i^d(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ . The key to the derivation of these probabilities is the use of a recursion at the slot boundaries of consecutive slots. The recursion is initiated for  $n=1$  with the following obvious relations:

$$P_i^d(j, 1) = \begin{cases} 1, & j=0, \\ 0, & j \geq 1, \end{cases} \quad i=0, 1, \dots, M-1, \quad (29)$$

and for  $i=M$  we have

$$P_M^d(j, 1) = \begin{cases} b_0, & j=0, \\ 1-b_0, & j=1. \end{cases} \quad (30)$$

For  $n \geq 2$ , we have the following recursive equations:

$$\begin{aligned}
P_i^d(0, n) &= \sum_{k=0}^{M-i} b_k P_{(i-1)^++k}^d(0, n-k) \\
&+ \mathbf{1}\{n \leq M-i\} \Pr\{b > n\}, \\
& \quad 0 \leq i \leq M, \\
P_i^d(j, n) &= \sum_{k=0}^{M-i} b_k P_{(i-1)^++k}^d(j, n-k) \\
&+ \sum_{k=1}^j b_{M-i+k} \\
& \quad \cdot P_{M-1\{i \neq 0\}}^d(j-k, n-(M-i+k)) \\
&+ \mathbf{1}\{n=M-i+j\} \Pr\{b > n\}, \\
& \quad 0 \leq i \leq M, \quad j \geq 0. \quad (31)
\end{aligned}$$

The explanation of (31) is clear and similar to the explanation of (27) and (28).

The probabilities  $P_i^d(j, n)$ ,  $0 \leq i \leq M$ ,  $n \geq 1$ ,  $0 \leq j \leq n$ , can be computed recursively in the parameter  $n$  from (31). In the first step, these probabilities are computed for  $n=1$  using the initial conditions in (29)–(30). In step  $k$ ,  $2 \leq k \leq n$ , the probabilities  $P_i^d(j, k)$  for any  $0 \leq j \leq k$  are computed for each  $i=0, 1, \dots, M$ , in increasing order. The number of simple operations needed in this procedure is of the order  $O(n^3 + M^2n)$ . The memory required for storing these probabilities throughout the procedure is  $n(n+3)(M+1)/2$ .

Once the probabilities  $P_i^d(j, k)$ ,  $0 \leq i \leq M$ ,  $1 \leq k \leq n$ ,  $0 \leq j \leq k$ , have been obtained, the probabilities  $P(j, n)$ ,  $0 \leq j \leq n$ , can be obtained directly from (25) and (27), (28).

*Numerical Example:* For the decoding scheme proposed in [15], a lost packet can be recovered, if and only if it is the only lost packet in its block. This is done by adding one parity packet to every block of size  $(n-1)$  packets, which increases the packet arrival rate to the system to  $\lambda(1 + (1/n - 1))$ . The average number of packets lost in a block after decoding is given by  $ED = \sum_{j=2}^n jP(j, n)$ , and the packet loss rate after decoding,  $P_{dec}$  is given by  $P_{dec} = ED/n$ . In Table IV, we compare the loss probability of a packet within a block of size  $n$  (before and after decoding), as given in [15] using the independence assumption and as computed in this section; for a system that can accommodate  $M=20$ , arrival rate  $\lambda=0.8$ , and for  $n=71, 20, 11, 9, 7$ .

It can be seen from Table IV that the exact probability  $P_{dec}$  is many orders of magnitude higher than  $P_{dec}$  under the independence assumption as obtained in [15], and is close to  $P_{dec}$  as obtained from simulations there. Also the packet loss ratio without decoding is only approximated in [15]. The packet loss ratio for the previous system without decoding was also computed for average arrival rate of  $\lambda=0.8$  and block sizes of  $n=70, 19, 10, 8, 6$ , and it is equal to  $6.0347 \cdot 10^{-5}$ . Note, that there is no instance where decoding reduces the loss probability. That is, the increase in the loss probability due to the increase in the packet arrival rate, caused by adding one parity packet to each block of size  $n-1$ , supercede the decrease in loss probability due to the forward error correction scheme.

TABLE IV  
PROBABILITY OF PACKET LOSS WITH  $M = 20$  AND  $\lambda = 0.8$ ; ALL QUANTITIES ARE FOR  $\lambda(1 + 1/n - 1)$

$n$	Before Decoding		After Decoding	
	Independence Assumption	Exact Analysis	Independence Assumption	Exact Analysis
71	$4.1728 \cdot 10^{-5}$	$9.3693 \cdot 10^{-5}$	$1.2171 \cdot 10^{-7}$	$8.0105 \cdot 10^{-5}$
20	$1.4972 \cdot 10^{-4}$	$2.9245 \cdot 10^{-4}$	$4.2528 \cdot 10^{-7}$	$2.3539 \cdot 10^{-4}$
11	$6.5533 \cdot 10^{-4}$	$1.0848 \cdot 10^{-3}$	$4.2805 \cdot 10^{-6}$	$8.2268 \cdot 10^{-4}$
9	$1.3547 \cdot 10^{-3}$	$2.0629 \cdot 10^{-3}$	$1.4601 \cdot 10^{-5}$	$1.5218 \cdot 10^{-3}$
7	$4.1127 \cdot 10^{-3}$	$5.5023 \cdot 10^{-3}$	$1.0024 \cdot 10^{-4}$	$3.8913 \cdot 10^{-3}$

## V. DISCUSSION

In this paper, we analyzed the distribution of the number of lost packets in fixed and variable size blocks of arrivals. We considered single and multiple session systems with Poisson and Bursty traffic models for each session.

The numerical results of the fixed-size single session system show that the use of an independence assumption for the rejection of packets from the system can lead to erroneous (many orders of magnitude lower than the exact results) evaluation of the above distribution. One has to be careful regarding the ability to cope with packet losses for real time applications such as voice and video since losses tend to concentrate in relatively short time intervals. It is known that applications like voice and video can tolerate a fair percentage of packet loss with no significant impact on the perceived quality. However, most tests and simulations were performed under independence loss assumption which tends to spread the losses uniformly [10]. For example, the results of Table I demonstrate that for a queue size of length  $M = 20$ , even under modest utilization (0.8) there are more blocks (of length  $n = 10$ ) that contain at least 2 losses ( $\sum_{j=2}^n P(j, n) = 6.11 \cdot 10^{-3}$ ) than there are blocks with a single packet loss ( $P(1, n) = 3.86 \cdot 10^{-3}$ ) where the rate of loss is only around 0.01 (an average of 0.1 lost packets per block). This can significantly impact the performance of smoothing and predictive playback algorithms. For similar reasons, (and using the same example) the independence assumption can also lead to wrong conclusions regarding the benefits of techniques such as forward error correction and makes them look far more useful than they really are. Such schemes have been proposed for both data and video communication. Note also that forward error correction increases the arrival rate of packets to the system due to the addition of parity packets, and hence increases the rejection probability of packets from the system as shown in Section IV.

For the fixed-size block, continuous-time single session system, we have obtained that the probability of no packet loss in a block ( $P(0, n)$ ) under the independence assumption can be substantially lower than the actual probability of no loss. These results demonstrate that buffer sizing under the independence assumption will be very pessimistic and the usefulness of a forward error correction scheme is even less attractive. For a fixed-size block, continuous-time multiple session system, we have shown that the correlation between lost packets of the same session  $s$  decreases as the rate of

session  $s$  decreases, for a given arrival rate to the system. However, even for relatively low rate sessions (such as 0.1 of the link capacity) the results are still quite far from the independence assumption approximation.

In what follows, we describe possible extensions of our model. The analysis of the packet loss process for the single session system as developed in Section II-A can be extended to the corresponding  $G/M/1/M$  model in a straightforward manner. The differences would be in the computation of the steady-state probabilities of the number of packets in the system at an arrival epoch (which is more complicated for a general arrival process), and of the number of packets served during an interarrival period (which is easily computed for exponential service time).

The analysis of the corresponding single session  $M/G/1/M$  model can be done by developing the recursion at *departure* epochs rather than at arrival epochs. Then, the analysis can follow in a similar manner to the single session system case in discrete time. The steady-state distribution of the number of packets in the system at arrival epochs is computed in [8]. Also, the multiple session  $M/G/1/M$  system can be analyzed correspondingly.

The same method used in this paper in order to obtain the distribution of the number of lost packets in a *block* can be applied to obtain the distribution of the number of lost packets in a *block of consecutive slots*, in the discrete-time model, or in a *continuous interval of time*, in the continuous time model.

We have omitted these extensions from the paper in order to avoid cumbersome calculations which do not add much to the understanding.

## APPENDIX A

Consider the following set of linear equations:

$$\begin{aligned}
 X_i &= a_i + \sum_{j=0}^{i+1} b_{i,j} X_j, \quad 0 \leq i \leq M-1, \\
 X_M &= a_M + \sum_{j=0}^M b_{M,j} X_j,
 \end{aligned} \tag{32}$$

where  $a_i, b_{i,j}, 0 \leq i \leq M, 0 \leq j \leq i+1$ , are given real numbers and  $X_i, 0 \leq i \leq M$ , are the unknown variables. Without loss of generality, we assume that  $b_{i,i+1}, 0 \leq i \leq M-1$ , have nonzero values. The set of linear equations in (32) defines the well known Hessenberg matrix [7].

Define  $X_i \triangleq \alpha_i + \beta_i X_0$ ,  $0 \leq i \leq M$ . In what follows, we describe a recursion for the calculation of the coefficients  $\alpha_i$ ,  $\beta_i$ ,  $0 \leq i \leq M$ , and the variable  $X_0$ .

$$\begin{aligned} \alpha_0 &= 0, & \beta_0 &= 1 \\ \alpha_i &= \frac{\alpha_{i-1} - \alpha_{i-1} - \sum_{j=0}^{i-1} b_{i-1,j} \alpha_j}{b_{i-1,i}}, \\ \beta_i &= \frac{\beta_{i-1} - \sum_{j=0}^{i-1} b_{i-1,j} \beta_j}{b_{i-1,i}}, & 1 \leq i \leq M, \\ X_0 &= \frac{a_M - \alpha_M + \sum_{j=0}^M b_{M,j} \alpha_j}{\beta_M - \sum_{j=0}^M b_{M,j} \beta_j}. \end{aligned}$$

The computation complexity of this procedure is of the order  $O(M^2)$ .

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